Sec. 5.2 Logarithms and Exponential Models

Exponential Law:

$$A = A_0 e^{kt}$$

where A is the amount after time t and A_0 is the original amount and k is a constant

Law of uninhibited growth – k > 0
Law of uninhibited decay – k < 0

Uninhibited Growth of Cells:

-A model that gives the number P of cells in the culture after a time t has passed (in the early stages of growth) is given by:

$$P(t) = P_0 e^{kt}, \ k > 0$$

where P_0 is the initial number of cells and k is a positive constant that represents the growth rate of the cells.

Ex. A colony of bacteria grows according to the law of uninhibited growth $P(t) = 100e^{.045t}$ where P is measured in grams and t is measured in days.

a. Determine the initial amount of bacteria. 100 grams

b. What is the growth rate of the bacteria? 4.5% continuous growth rate

c. What is the population after 5 days?

$P(5) = 100e^{.045(5)}$

$= 125.232$ grams

d. How long will it take for the population to reach 140 grams?

$\frac{140}{100} = \frac{100e^{.045t}}{100}$

$\frac{7}{5} = e^{.045t}$

$\ln\left(\frac{7}{5}\right) = \ln e^{.045t}$

$\frac{\ln\left(\frac{7}{5}\right)}{.045} = \frac{.045t}{.045}$

7.477 days $= t$

e. What is the doubling time for the population?

$\frac{200}{100} = \frac{100e^{.045t}}{100}$

$2 = e^{.045t}$

$\ln 2 = \ln e^{.045t}$

$\frac{\ln 2}{.045} = \frac{.045t}{.045}$

15.403 days $= t$

Ex. A colony of bacteria increases according to the law of uninhibited growth.

a. If the number of bacteria doubles in 3 hours, find the function that gives the number of cells in the culture.

$P = P_0 e^{kt}$

$2P_0 = P_0 e^{k3}$

$2 = e^{3k}$

$\ln 2 = \ln e^{3k}$

$\frac{\ln 2}{3} = \frac{3k}{3}$

$k = .23105$

$\boxed{P = P_0 e^{.23105t}}$

b. How long will it take for the size of the colony to triple?

$3P_0 = P_0 e^{.23105t}$

$3 = e^{.23105t}$

$\ln 3 = \ln e^{.23105t}$

$\frac{\ln 3}{.23105} = \frac{.23105t}{.23105}$

$\boxed{4.755 \text{ hours} = t}$

c. How long will it take for the size of the colony to double a second time (quadruple)?

$4P_0 = P_0 e^{.23105t}$

$4 = e^{.23105t}$

$\ln 4 = \ln e^{.23105t}$

$\frac{\ln 4}{.23105} = \frac{.23105t}{.23105}$

$\boxed{6.000 \text{ h} = t}$

Uninhibited Radioactive Decay:

The amount A of a radioactive material present at time t is given by the following model:

$$A(t) = A_0 e^{kt}, \ k < 0$$

where A_0 is the original amount of radioactive material and k is a negative number that represents the rate of decay.

Ex. Traces of burned wood along with ancient stone tools in an archaeological dig in Chile were found to contain approximately 1.67% of the original amount of carbon 14. If the half-life of carbon 14 is 5600 years, approximately when was the tree cut and burned?

a. Solve for k using the half-life in order to find the exponential model.

$A(t) = A_0 e^{kt}$

$\frac{1}{2}A_0 = A_0 e^{k(5600)}$

$\frac{1}{2} = e^{5600k}$

$\ln(\frac{1}{2}) = \ln e^{5600k}$

$\frac{\ln(\frac{1}{2})}{5600} = \frac{5600k}{5600}$

$k = -.000123776$

$A(t) = A_0 e^{-.000123776t}$

b. Use that model to determine the time in which the given percentage of the original amount remained.

$A(t) = A_0 e^{-.000123776t}$

$.0167A_0 = A_0 e^{-.000123776t}$

$.0167 = e^{-.000123776t}$

$\ln(.0167) = \ln(e^{-.000123776t})$

$\frac{\ln(.0167)}{-.000123776} = \frac{-.000123776t}{-.000123776}$

$\boxed{33{,}062 \text{ years} = t}$

Ex. The US population, P, in millions, is currently growing according to the formula:

$$P = 299e^{0.009t}$$

where t is the years since 2006. When is the population predicted to reach 350 million?

$\frac{350}{299} = \frac{299e^{.009t}}{299}$

$\frac{350}{299} = e^{.009t}$

$\ln\left(\frac{350}{299}\right) = \ln\left(e^{.009t}\right)$

$\frac{\ln\left(\frac{350}{299}\right)}{.009} = \frac{.009t}{.009}$

$\boxed{17.499y = t}$

Dec 31, 2007 – 1 year since 2006
Dec 31, 2008 – 2 years since 2006
Dec 31, 2023 – 17 years since 2006
Dec 31, 2024 – 18 years since 2006

During the year 2024.

Converting between $Q = ab^t$ and $Q = ae^{kt}$:

Ex. Convert the exponential function $P = 175(1.145)^t$ to the form $P = ae^{kt}$.

$ae^{kt} = 175(1.145)^t$ $\quad a = 175$ when $t = 0$

$e^{kt} = 1.145^t$

$(e^k)^t = (1.145)^t$

$e^k = 1.145$ $\quad K = .1354$ $\quad \boxed{P = 175e^{.1354t}}$

$\ln e^k = \ln 1.145$

$K = \ln 1.145$

Ex. Convert the formula $Q = 7e^{0.3t}$ to the form $Q = ab^t$:

$Q = 7(e^{.3})^t$

$Q = 7(1.3499)^t$

Ex. Find the continuous and annual growth rates of the previous two examples.

EX 1: CONTINUOUS GROWTH RATE: 13.54%
ANNUAL GROWTH RATE: 14.5%

EX 2: CONTINUOUS GROWTH RATE: 30%
ANNUAL GROWTH RATE: 34.99%

Ex. Find the continuous percent growth rate of $Q = 200(0.886)^t$

$.886^t = e^{kt}$

$.886^t = (e^k)^t$

$.886 = e^k$

$\ln(.886) = \ln e^k$

$\ln(.886) = K$

$-.12104 = K$

CONTINUOUS GROWTH RATE: $\boxed{-12.104\%}$

HW: pg 194-197 #3-57 (m/3)